Gibbsian Hypothesis in Turbulence

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Abstract

We show that Kolmogorov multipliers in turbulence cannot be statistically independent of others at adjacent scales (or even a finite range apart) by numerical simulation of a shell model and by theory. As the simplest generalization of independent distributions, we suppose that the steady-state statistics of multipliers in the shell model are given by a translation-invariant Gibbs measure with a short-range potential, when expressed in terms of suitable "spin" variables: real-valued spins that are logarithms of multipliers and XY-spins defined by local dynamical phases. Numerical evidence is presented in favor of the hypothesis for the shell model, in particular novel scaling laws and derivative relations predicted by the existence of a thermodynamic limit. The Gibbs measure appears to be in a high-temperature, unique-phase regime with "paramagnetic" spin order.

1 Introduction

In a famous paper on the local structure of turbulence of incompressible fluids [1], A. N. Kolmogorov in 1962 considered inertial-range multipliers defined by ratios of velocity increments $w_{ij}(\ell,\ell') = \delta_{i,\ell}v_j/\delta_{i,\ell'}v_j$. Here $\delta_{i,\ell}v_j(\mathbf{x}) = v_j(\mathbf{x}+\ell\mathbf{e}_i) - v_j(\mathbf{x})$ is the increment of the jth component of the velocity vector $\mathbf{v}$ along the unit vector $\mathbf{e}_i$. Kolmogorov hypothesized that, at very high Reynolds number, these multipliers should have distributions which are universal functions only of the scale-ratio ℓ/ℓ' and not of the absolute scale. He postulated further that multipliers corresponding to widely separated scales should be statistically independent. See also [2]. R. Benzi, L. Biferale, and G. Parisi [3] have made a more precise and quite remarkable hypothesis in the context of a "shell model" of turbulence. The latter are quadratically nonlinear dynamical systems for variables in a finite number N of shells with wavenumbers $k_n = \lambda^n k_0$, $n = 1, ..., N$ with $\lambda > 1$. For example, the SABRA model [4] has one complex mode u_n per shell which obeys a dynamical equation

$$\begin{aligned}\frac{du_n}{dt} &= i(ak_{n+1}u_{n+2}u^*_{n+1} + bk_n u_{n+1}u^*_{n-1} \\ &\quad -ck_{n-1}u_{n-1}u_{n-2}) - \nu k_n^2 u_n + f_n. \qquad (1)\end{aligned}$$

([3] considered a slightly different model.) Here $*$ stands for complex conjugation, f_n is a forcing term which is restricted to the first few shells and ν is the "viscosity". When $a+b+c=0$, equation (1) satisfies conservation of "energy" $E = \frac{1}{2}\sum_n |u_n|^2$ and "helicity" $H = \sum_n (a/c)^n |u_n|^2$ in the $\nu \to 0$ limit, analogous to the quadratic invariants of the inviscid Euler equations. Then, for large N and high Reynolds number $Re = \sqrt{\langle |u_1|^2\rangle}/(\nu k_0)$, the authors of [3] hypothesized that the variables u_n, $n = 1, ..., N$ should have a steady-state statistical distribution given by a *Gibbs measure*

$$P(u_1, ..., u_N) \propto \exp[-\sum_n \Phi_n(u_n, u_{n-1}, u_{n+1}, ...)]. \qquad (2)$$

Because turbulence is a dissipative state, far from thermodynamic equilibrium, the potentials Φ_n have nothing to do with the inviscid invariants, even as $\nu \to 0$. For shellnumbers n in the long inertial range, $1 \ll n \ll N$, [3] supposed that the potential Φ_n becomes a universal function Φ independent of n. They supposed further that the potential Φ is a sufficiently short-range function of "the ratios between the u's and their angles" [3]. The authors of [3] used an "infinite-temperature" model with independent multipliers to predict the scaling exponents. It is usually assumed that such an approximation is qualitatively correct. However, we show that the multipliers cannot be strictly independent and that the correlations are essential. We also give evidence for the Gibbs hypothesis, based on theoretical analysis and direct numerical simulation of the shell-model dynamics (1).

2 Theoretical Considerations

First, let us give a more precise form to the hypothesis. In the shell model, we introduce an amplitude ρ_n and a phase θ_n for each shellnumber n, via the polar decomposition $u_n = k_n^{-1/3} \cdot \rho_n e^{i\theta_n}$. Following [3], we separate out the Kolmogorov 1941 scaling factor $k_n^{-1/3}$. If $w_n = \rho_n/\rho_{n-1}$ is the multiplier defined by [3] and $\Delta_n = -\theta_n + \theta_{n-1} + \theta_{n-2}$ is the dynamical phase factor [3, 4], then we define

$$\sigma_n = \ln(w_n),\ U_n = \exp(i\Delta_n) \tag{3}$$

where σ_n is a local "slope" and U_n is an "XY-spin" or 2-dimensional rotator spin. The definition of σ_n is motivated by the observation that $\ln\rho_n = \sum_{k=1}^{n}\sigma_k$ is then a "total spin". Alternatively, $h_n = \ln\rho_n$ can be viewed as a "height function", as in equilibrium models of surface roughness. Then, our hypothesis is that the distribution of these "spin variables" $\xi_n = (\sigma_n, U_n)$ is a translation-invariant Gibbs measure with a short-range potential Φ, ignoring finite-size effects from the forcing and dissipation ranges of shellnumber n. The potential is expected to be, at least, absolutely summable ([5], section 2.1), which guarantees its uniqueness up to physical equivalence. However, the numerical evidence presented below suggests that the interactions are not merely summable, but indeed quite rapidly decaying.

On the other hand, the potentials cannot have a strictly finite range R, i.e. vanishing for any set of spin variables containing pairs $\xi_n, \xi_{n'}$ with $|n - n'| > R$. In particular, the assumption of zero-range interactions, $R = 0$, or independent spins which was made made by [3] is ruled out. Using exact constraints from the dynamics, we show in Appendix 1 that the assumption of independent spins leads to the deterministic K41 fixed-point $u_n \sim -i(\varepsilon/k_n)^{1/3}$ as the only statistically stable solution. In reality this solution is dynamically unstable [7]. The false assumption of independence stabilizes this solution and prevents intermittency corrections from developing. This is plausible, since intermittency is known to arise in the shell models from "burst" solutions which exhibit long-range coherence in both σ_n and U_n over many shells (e.g. see Fig.4 in [8]). Furthermore, the K41 fixed-point can be shown to be stabilized by assuming *any* finite-range potential between spins [6], so that the stationary measure cannot be Gibbs with any potential of strictly finite-range.

Another argument against independence of spins is that this would imply the quadratic equation

$$a\mu_3^2 + b\mu_3 + c = 0 \tag{4}$$

for the multiplier moment $\mu_p = \langle w^p \rangle,\ p = 3$. Cf. [3], Eq.(18). This is analogous to the "4/5-law" of fluid turbulence . Note that, in the independent spin approximation, the structure-function scaling exponents are given by $\zeta_p = (p/3) - \log_\lambda \mu_p$. There are two roots of the quadratic equation, $\mu_3 = 1$ and $\mu_3 = c/a$. The first solution gives constant mean energy flux but zero

helicity flux, while the second gives the opposite. Thus, no joint cascade of energy and helicity is possible in the independent spin approximation, contrary to observations [9]. Furthermore, the second solution violates the realizability inequality $\mu_3 \geq 0$, when $c/a < 0$ and the second invariant is truly "helicity-like". These conclusions do not remain true for a potential of finite but non-zero range. The quadratic equation still holds for a nearest-neighbor potential or Markov chain approximation, where now μ_3 is the principal eigenvalue of a "transfer matrix" [6]. As shown in Appendix 2 of the present paper, joint cascades of energy and helicity are permitted in a Markov chain model, if $\mu_3 = 1$ and the subleading eigenvalue $\mu_3' = c/a$. Furthermore, a concrete Markov chain model is constructed in the appendix to show that this situation may be realized. While the Gibbs measure cannot be exactly nearest-neighbor, we believe that this may be a good working approximation.

3 Numerical Results

We now present our simulation results for the SABRA model, with standard choice of parameters $\lambda = 2, k_0 = 2^{-4}, a = 1, b = c = -1/2$. We performed two sets of simulations, one with $N = 22$, $\nu = 10^{-7}$ and the second with $N = 26$, $\nu = 2 \cdot 10^{-9}$. A force $f_n = F_n(1+i)/u_n^*$ was used with F_n real, nonzero only for $n = 2, 3$, and chosen to give an input of energy but not helicity. Except where stated, the results shown are for the $N = 26$ simulation. Stationary time-averages were achieved by integrating over a period of more than 2900 large-eddy turnover times. In Fig. 1 are shown the energy spectrum and mean energy flux in the simulation.

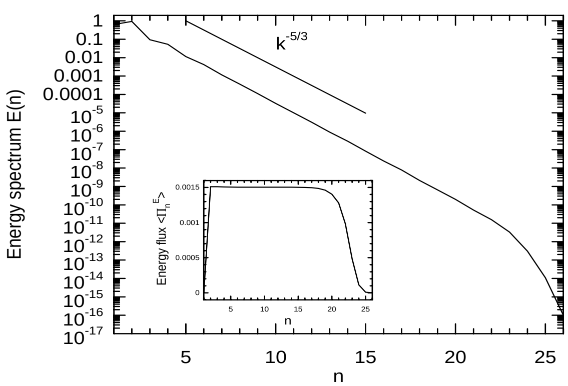

FIG. 1. Energy spectrum shows $k^{-5/3}$ power law from 5th shell to 23rd shell. In the inset we show a constant mean energy flux in the inertial range.

We first present evidence for the good decay of correlations of "spins". We define the spin-spin correlation functions $C_{XY}(n;m) = \langle X_n^* Y_m \rangle - \langle X_n^* \rangle \langle Y_m \rangle$ for $X, Y = \sigma, U$. The results are shown in Figs. 2-3. It may be seen that the correlations decay quite rapidly in $|m-n|$, exponentially or as a large inverse power (≥ 7). This is an indication that the Gibbs measure of the hypothesis does not correspond to a critical point with a power-law scaling. For the

"1-dimensional" spin chain of the shell model a phase-transition would, in any case, require a long-range potential Φ, e.g. a pair interaction decaying by a small inverse power ≤ 2 (e.g. see [10]). The hypothesis of a short-range potential therefore rules out any such critical behavior.

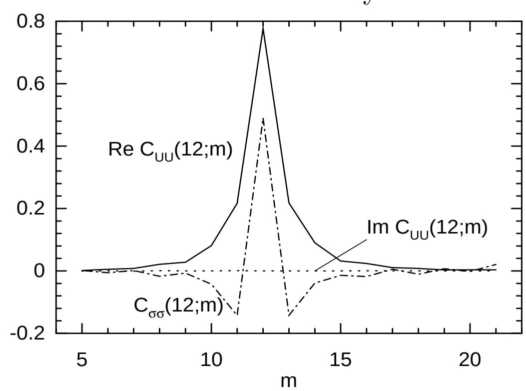

FIG. 2. $C_{\sigma\sigma}(n;m)$, $Re\ C_{UU}(n;m)$ and $Im\ C_{UU}(n;m)$ for n=12.

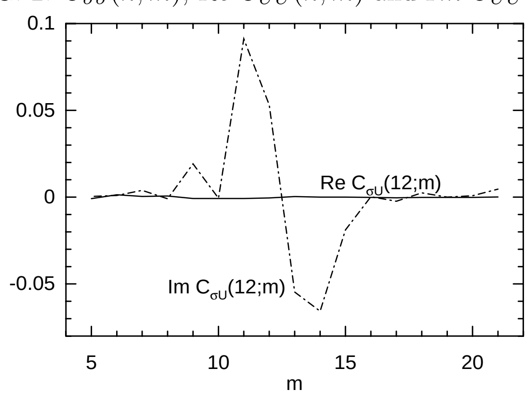

FIG. 3. $Re\ C_{\sigma U}(n;m)$ and $Im\ C_{\sigma U}(n;m)$ for n=12.

Further evidence against any critical behavior comes from a consideration of the thermodynamics. The Gibbs hypothesis implies that a "Gibbs free-energy" g should exist for the spin models, defined by a suitable "thermodynamic limit". Thus, we may introduce "magnetic fields" p corresponding to the σ-spins and $h = h_x + ih_y$ corresponding to the U-spins, so that the concave free-energy is defined by

$$g(p,h) = \lim_{n\to\infty} \frac{-1}{n} \ln Z_n(p,h) \tag{5}$$

with the "partition function"

$$Z_n(p,h) := \langle \exp \left[\sum_{k=1}^{n} (p\sigma_k + \mathrm{Re}(h^* U_k)) \right] \rangle. \tag{6}$$

The absolute structure functions are proportional to the "partition functions" at $h = 0$: $\langle |u_n|^p \rangle = k_n^{-p/3} Z_n(p,0)$. Thus, the existence of a thermodynamic limit, as implied by the Gibbs hypothesis, yields a power-law scaling of

the structure functions $\sim k_n^{-\zeta_p}$ with the anomalous exponent $\delta\zeta_p := \zeta_p - \frac{p}{3}$ related to the free-energy by $g(p,0) = \delta\zeta_p \cdot \ln\lambda$. However, the Gibbs hypothesis implies also a power-law scaling for the "phase structure-functions" $Z_n(0,h)$. Fig. 4 shows clean power-law ranges for these quantities, solid evidence in favor of the Gibbs hypothesis (but not a proof, since the thermodynamic

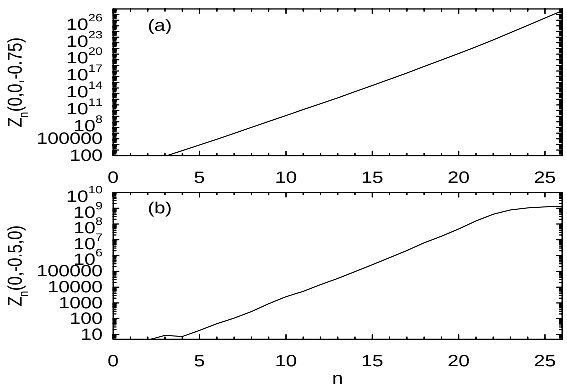

FIG. 4. Phase structure functions (a) $Z_n(0,0,-0.75)$; (b) $Z_n(0,-0.5,0)$.

limit could exist even for a non-Gibbsian measure; e.g see [11]). In Fig. 5 we plot cross-sections of the Gibbs free energy, $g(0,h_x,0)$ and $g(0,0,h_y)$. These appear to be smooth functions of their arguments. There is no evidence for any non-analyticity that would signal appearance of a phase transition.

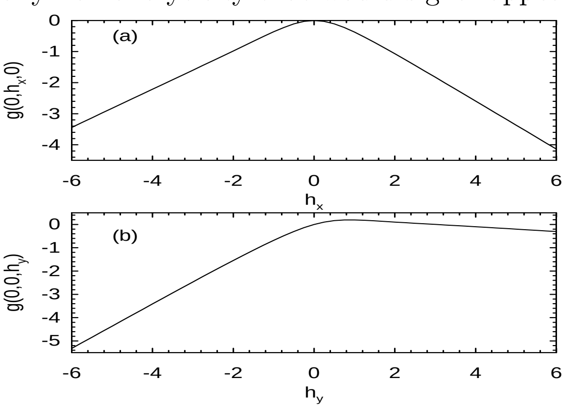

FIG. 5. Gibbs free energy (a) $g(0,h_x,0)$; (b) $g(0,0,h_y)$.

We now study the question of "translation-invariance" of the measures, by means of the single-site spin distributions. We plot in Figs. 6 and 7 the distributions $P(\sigma_n)$ and $P(\Delta_n)$, for different values of n in the inertial range. We see that these distributions collapse quite well, verifying the "translation-invariance" assumption. The first distribution is approximately exponential type $P(\sigma) \approx (\alpha/2)\exp(-\alpha|\sigma|)$ with $\alpha \approx 2.00$ while the second is fit well by $P(\Delta) \approx C\exp(-\beta\sin(\Delta))$ with $C \approx 0.1, \beta \approx 1.2$. The "infinite-temperature" model in [3] does not predict well either $P(\sigma)$ or $P(\Delta)$. That approximation assumed a distribution $P(\Delta)$ uniform on the interval $[-\pi, 0]$ and yielded a multiplier distribution $P(w)$ also compactly supported on a finite interval $[w_-, w_+]$. However, the result in Fig. 6 (see inset) implies a distribution $P(w)$ with two power-law regimes, $\sim w^{\alpha-1}$ for $w \ll 1$ and

$\sim w^{-\alpha-1}$ for $w \gg 1$. This is inconsistent not only with [3] but with any independent spin model. Because of the power-law tail for large w, the moments which would give the anomalous scaling exponents for independent spins in fact diverge, $\langle w^p \rangle = +\infty$ for $p \geq \alpha$.

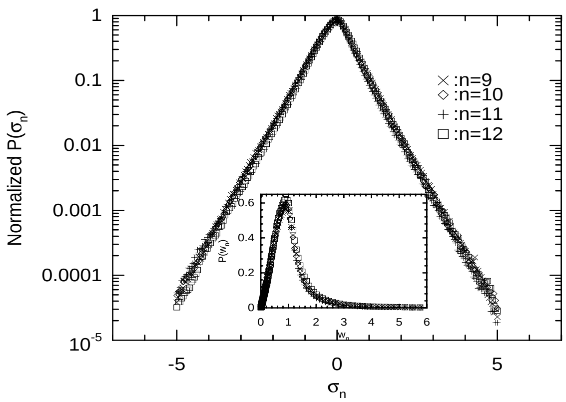

FIG. 6. Distributions of σ_n, for $n = 9 - 12$. In the inset are the distributions of w_n.

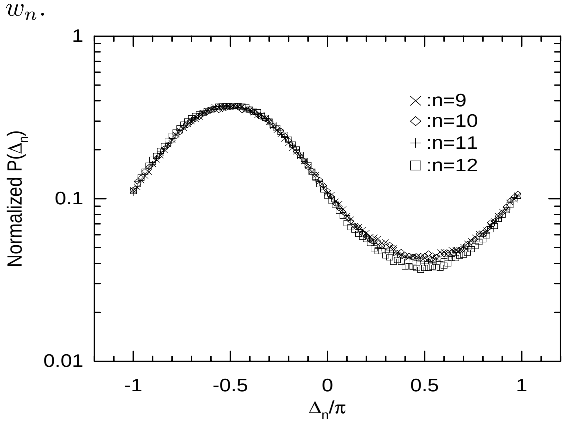

FIG. 7. Distributions of Δ_n, for $n = 9 - 12$.

The distributions for the σ-spins appear symmetric under spin-flip $\sigma \to -\sigma$ around the K41 value $\sigma = 0$. In fact, the exponential tails on the right and the left arise symmetrically from the same events in SABRA, which we call "defects". These are events in which the amplitude in a single shell, say, the nth, drops to a very low value, $\rho_n \ll 1$. Intermittent bursts in the shell models are generally preceded by such events [8, 12], but defects do not need to appear in association with bursts. Since $\sigma_n = \ln(\rho_n/\rho_{n-1})$, the negative tail where $\sigma_n \ll -1$ comes from realizations with a "defect" in the nth shell, while the positive tail where $\sigma_n \gg 1$ comes from realizations with "defects" in the $(n-1)$st shell. The statistical distribution of such "defects" in a given shell n can be inferred from the constancy of $P(u_n)$ at $u_n = 0$. By a change to polar coordinates (ρ_n, θ_n), one finds $P(\rho_n) \approx (const.)\rho_n$ for $\rho_n \ll 1$. For events in the negative tail, $\rho_n \ll 1$ while $\rho_{n-1} \approx 1$. In that case, $\sigma_n \approx \ln \rho_n$ and $\rho_n \approx e^{\sigma_n}$. Therefore, by change of variables, $P(\sigma_n) = \left|\frac{d(\rho_n)}{d\sigma_n}\right| \cdot P(\rho_n)\Big|_{\rho_n \approx e^{\sigma_n}} \approx (const.)e^{2\sigma_n}$ for $\sigma_n \ll -1$. An identical

argument for the positive tail using $\sigma_n \approx -\ln(\rho_{n-1})$ and $\rho_{n-1} \approx e^{-\sigma_n}$ gives $P(\sigma_n) \approx (const.)e^{-2\sigma_n}$ for $\sigma_n \gg 1$. This argument assumes only that the amplitudes ρ_n, ρ_{n-1} in the ratio are not strongly correlated, which could suppress the long tails. The distribution $P(\Delta)$ also exhibits a symmetry $\Delta \to \pi - \Delta$ around the K41 solution with $\Delta = -\pi/2$. This symmetry can be expressed as $U \to -U^*$ and is seen as well in the vanishing of $\mathrm{Im}\, C_{UU}$ and $\mathrm{Re}\, C_{\sigma U}$ in Figs. 2-3 and in the near symmetry of $g(0, h_x, 0)$ in Fig. 5. In fact, this is an exact symmetry of the dynamics, broken only by our forcing. If u_n is a solution of SABRA with a force f_n, then $-u_n^*$ is a solution with force $-f_n^*$ and under this transformation $U_n \to -U_n^*$ for all n. This symmetry should be restored for large n, similar to restoration of isotropy in 3D.

The distribution $P(\sigma)$ cannot be exactly symmetrical under the spin-flip $\sigma \to -\sigma$. There must be a non-vanishing mean $\langle\sigma\rangle$ or a "magnetization", due to the fact that Kolmogorov 1941 mean-field scaling of pth-order structure functions is not exact at $p = 0$ [13]. Indeed, by the Gibbs hypothesis, the magnetization can be obtained from the thermodynamic formula $\langle\sigma\rangle = -\left.\frac{\partial g}{\partial p}\right|_{p,h=0}$. In Fig. 8 we plot $\langle\sigma_n\rangle$ vs. n for the two simulations of SABRA

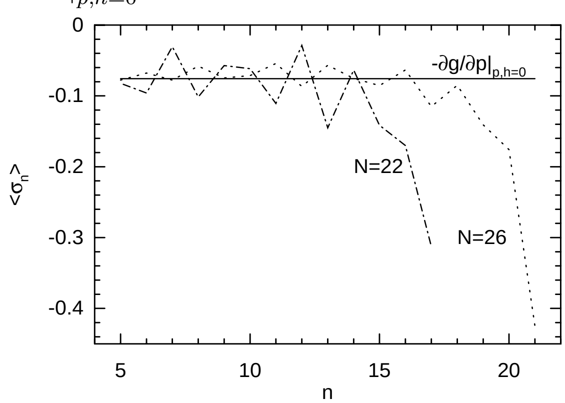

FIG. 8. $\langle\sigma_n\rangle$ vs. n with $-\left.\partial g/\partial p\right|_{p,h=0}$.

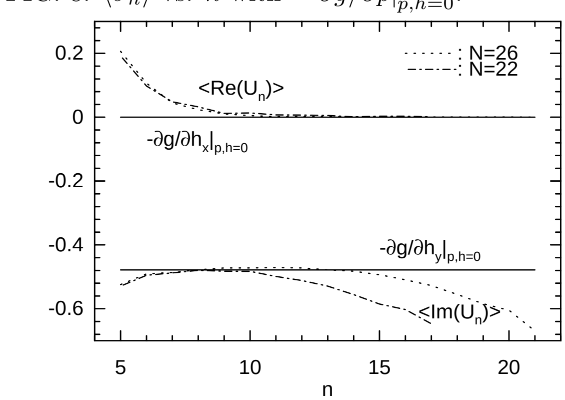

FIG. 9. $\langle Re(U_n)\rangle$ vs. n with $-\left.\partial g/\partial h_x\right|_{p,h=0}$ and of $\langle Im(U_n)\rangle$ vs. n with $-\left.\partial g/\partial h_y\right|_{p,h=0}$.

with $N = 22$ and $N = 26$. We see that there is a slight breaking of translation symmetry. This is likely due to finite-size effects and becomes much

smaller for the $N = 26$ shell simulation than for $N = 22$. In the same figure is plotted a straight horizontal line for the prediction $\langle\sigma\rangle \approx -0.0757$ obtained from the derivative of g. We see that the agreement is quite satisfactory. It is also apparent from Fig. 7 that a non-zero expectation $\langle U\rangle$ occurs, non-invariant under conjugation $U \to U^*$. In fact, a negative value $\langle\sin(\Delta)\rangle < 0$ is required in SABRA for a forward cascade of energy [3, 4]. We plot in Fig. 9 the "magnetizations" $\langle U_n\rangle$ vs. wavenumber n. Again we see a finite-size breaking of translation-symmetry, which lessens going from $N = 22$ to $N = 26$. The numerical values of these "magnetizations" are also predicted by derivatives of the Gibbs free energy, giving a large expectation $\langle\sin\Delta\rangle \approx -0.489$ but a small value $\langle\cos\Delta\rangle = 0.00814$, consistent with approximate symmetry $U \to -U^*$. These results are plotted in Figs. 8 and 9 as horizontal lines and obviously give satisfactory agreement with the direct measurements of magnetizations. The results indicate that there is "spin-ordering" in the turbulent systems, which breaks discrete spin-flip symmetries. Peierls-type arguments [14], including rigorous versions such as Pirogov-Sinai theory [15], indicate that discrete symmetries cannot be spontaneously broken in "1-dimensional lattice" systems such as the shell model, if the interaction potential is short-ranged. A 1-dimensional Gibbs distribution with non-zero magnetization for a symmetric, short-ranged potential is unstable to formation of domain walls. We expect the spin order here to be not "ferromagnetic" but "paramagnetic", arising from explicit symmetry-breaking terms in the potential Φ of the Gibbs measure.

4 Discussion and Conclusions

In this work, we have shown that Kolmogorov multipliers and dynamical phases in the shell model must be correlated from shell-to-shell. We have also presented evidence that these "spin" variables, while not independent, are distributed according to a translation-invariant Gibbs measure. Of course, this has certainly not been proved in this work. There are non-Gibbsian measures on one-dimensional lattices, such as that of Schonmann [16], which have both exponential decay of correlations and a thermodynamic limit of the Gibbs free energy (which, however, is there non-analytic). On the other hand, that type of example has been shown to be Gibbs also in a somewhat generalized sense [17]. The hypothesis that the distribution of suitably defined "spins" in the shell model is Gibbsian with a summable potential has testable consequences, some of which we have verified in this work.

If the hypothesis is true, then it is possible to recover the potentials from the finite-shell marginal distributions. Indeed, if $P_{n,\dots,n+N-1}(\xi_n,\dots,\xi_{n+N-1})$ is the probability density of the "spins" at shells $n,\dots,n+N-1$ in the inertial range, then

$$\ln P_{n,\dots,n+N-1}(\xi_n,\dots,\xi_{n+N-1}) = -\sum_{k=0}^{N-1} \Phi_{n+k}(\xi) + o(N), \tag{7}$$

whenever the potential is absolutely summable. See Proposition 2.46 in [18]. More directly, the potentials may be derived from conditional probabilities of the spins via the Möbius inversion formula [19]. For example, if the "spins" are distributed by a Markov chain or nearest-neighbor Gibbs measure, then, up to constants, the 1-body interaction is

$$\Phi_n^{(1)}(\xi_n) = -\ln T_{n|n-1}(\xi_n|\xi^*) - \ln T_{n+1|n}(\xi^*|\xi_n) \tag{8}$$

and the 2-body interaction is

$$\begin{aligned}\Phi_{n,n-1}^{(2)}(\xi_n, \xi_{n-1}) \quad &= \quad -\ln T_{n|n-1}(\xi_n|\xi_{n-1}) \\ &\qquad + \ln T_{n|n-1}(\xi_n|\xi^*) + \ln T_{n|n-1}(\xi^*|\xi_{n-1}).\end{aligned} \tag{9}$$

Here $T_{n|n-1}(\xi_n|\xi_{n-1})$ is the transition probability of the Markov chain and ξ^* is a constant reference value of the spin. As this example makes clear, the potentials of the Gibbs measure can be recovered, in principle, from conditional probabilities obtained in numerical simulations of the dynamics.

If true, it is intellectually interesting that probability measures arising from turbulent dynamics may be Gibbsian, in a suitable spin representation. However, more importantly, it gives one some new tools that one may apply to the turbulence problem. For example, it offers a new route to calculate the scaling exponents as a "free energy" $g(p, h)$ of a one-dimensional spin system. Furthermore, formulas for the conditional probabilities of small-scale modes given the large-scale ones provided by the Gibbs hypothesis may be very useful in carrying out large-eddy simulations of turbulence [20]. Some preliminary tests of these ideas have already been carried out for the shell models and will be reported elsewhere [6].

Similar results as discussed here hold for Navier-Stokes dynamics [21], using the velocity increments advocated by Kolmogorov [1]. Multipliers and spins may also be defined by a representation of the Navier-Stokes equations in terms of orthogonal wavelet bases [22]. In that formulation, the dynamics resembles the shell model on the dyadic Cayley tree [23]. Generalizing [3], the invariant measure of the latter should be a Gibbs measure on a Bethe lattice ([24], Ch.4.) Whereas standard shell models corespond to "1-dimensional" spin systems, the spin systems on the Bethe lattice are effectively "infinite-dimensional". Nevertheless, the statistics in the shell model and in Navier-Stokes should be qualitatively similar, as present evidence suggests that the Gibbs distributions for both are in the high-temperature, unique-phase regime.

A Appendices

A.1 Stabilization of the K41 Solution

We show in this first appendix that the K41 solution would be stable if multipliers were independent for distinct shells (or, in fact, even if they

were independent for shells a finite distance apart). We show that this follows from a set of exact dynamical constraints on the multiplier variables $\xi_n = (w_n, \Delta_n)$.

For this purpose, we must transform the equations of motion into those variables. In terms of ρ_n, θ_n, the SABRA dynamics becomes

$$\begin{aligned}\frac{\dot{\rho}_n}{\rho_n} &= \frac{k_n^{2/3}}{\lambda}\left[a\frac{\rho_{n+2}\rho_{n+1}}{\rho_n}\sin\Delta_{n+2} + b\frac{\rho_{n+1}\rho_{n-1}}{\rho_n}\sin\Delta_{n+1} + c\frac{\rho_{n-1}\rho_{n-2}}{\rho_n}\sin\Delta_n\right] \\ &\quad + \frac{k_n^{1/3}\mathrm{Re}(f_n e^{-i\theta_n})}{\rho_n} - \nu k_n^2 \end{aligned} \tag{10}$$

and

$$\begin{aligned}\dot{\theta}_n &= \frac{k_n^{2/3}}{\lambda}\left[a\frac{\rho_{n+2}\rho_{n+1}}{\rho_n}\cos\Delta_{n+2} + b\frac{\rho_{n+1}\rho_{n-1}}{\rho_n}\cos\Delta_{n+1} - c\frac{\rho_{n-1}\rho_{n-2}}{\rho_n}\cos\Delta_n\right] \\ &\quad + \frac{k_n^{1/3}\mathrm{Im}(f_n e^{-i\theta_n})}{\rho_n}. \end{aligned} \tag{11}$$

Going over to the scale-local variables w_n, Δ_n, this becomes

$$\dot{w}_n = w_n[U_n(w,\Delta) - U_{n-1}(w,\Delta)] := W_n(w,\Delta)$$

and

$$\dot{\Delta}_n = -V_n(w,\Delta) + V_{n-1}(w,\Delta) + V_{n-2}(w,\Delta) := Z_n(w,\Delta)$$

where U_n, V_n are the righthand sides of (10),(11), respectively, expressed in terms of w_n, Δ_n. Thus, considering just the inertial-range part of the dynamics,

$$\begin{aligned}U_n(w,\Delta) &= \frac{k_n^{2/3}}{\lambda}\cdot\left[a\cdot w_{n+2}\sin\Delta_{n+2}\cdot w_{n+1}^2 w_n w_{n-1}\right. \\ &\quad \left. + b\cdot w_{n+1}\sin\Delta_{n+1}\cdot w_{n-1} + c w_n^{-1}\sin\Delta_n\right]\prod_{k=1}^{n-2} w_k \end{aligned} \tag{12}$$

and

$$\begin{aligned}V_n(w,\Delta) &= \frac{k_n^{2/3}}{\lambda}\cdot\left[a\cdot w_{n+2}\cos\Delta_{n+2}\cdot w_{n+1}^2 w_n w_{n-1}\right. \\ &\quad \left. + b\cdot w_{n+1}\cos\Delta_{n+1}\cdot w_{n-1} - c w_n^{-1}\cos\Delta_n\right]\prod_{k=1}^{n-2} w_k \end{aligned} \tag{13}$$

In terms of these variables, the dynamics appears highly nonlocal in scale, because of the product $\prod_{k=1}^{n-2} w_k$.

If $P_n(\xi_n)$ is the distribution of $\xi_n = (w_n, \Delta_n)$, then it is straightforward to show that

$$\partial_t P_n(\xi_n) = -\frac{\partial}{\partial w_n}\left[\langle W_n|\xi_n\rangle P_n(\xi_n)\right] - \frac{\partial}{\partial \Delta_n}\left[\langle Z_n|\xi_n\rangle P_n(\xi_n)\right], \tag{14}$$

where $\langle W_n|\xi_n\rangle, \langle Z_n|\xi_n\rangle$ are conditional averages for fixed ξ_n. These equations are exact, but not closed in terms of P_n. However, if we assume that the statistics of the model are given by a product measure $\prod_n P_n(\xi_n)$—and with *only* that assumption—then we obtain closed equations for the P_n's. In the inertial range, these equations are of the form (14) with

$$\langle W_n|\xi_n\rangle = \frac{k_n^{2/3}}{\lambda}\prod_{k=1}^{n-4}\langle w_k\rangle\left[(A_n w_n^3 - B_n w_n^2 + C_n w_n) + (D_n w_n^2 - E_n)\sin\Delta_n\right]$$

$$\langle Z_n|\xi_n\rangle = \frac{k_n^{2/3}}{\lambda}\prod_{k=1}^{n-4}\langle w_k\rangle\left[(F_n w_n^2 + G_n w_n + H_n) - (K_n w_n^{-1} + L_n w_n)\cos\Delta_n\right]$$

where

$$A_n = -\frac{a}{\lambda^{2/3}}\langle w_{n+1}S_{n+1}\rangle\langle w_{n-1}\rangle\langle w_{n-2}\rangle\langle w_{n-3}\rangle$$

$$B_n = -a\langle w_{n+2}S_{n+2}\rangle\langle w_{n+1}^2\rangle\langle w_{n-1}\rangle\langle w_{n-2}\rangle\langle w_{n-3}\rangle$$

$$C_n = b\langle w_{n+1}S_{n+1}\rangle\langle w_{n-1}\rangle\langle w_{n-2}\rangle\langle w_{n-3}\rangle - \frac{c}{\lambda^{2/3}}\langle\frac{S_{n-1}}{w_{n-1}}\rangle\langle w_{n-3}\rangle$$

$$D_n = -\frac{b}{\lambda^{2/3}}\langle w_{n-2}\rangle\langle w_{n-3}\rangle$$

$$E_n = -c\langle w_{n-2}\rangle\langle w_{n-3}\rangle$$

$$F_n = \frac{a}{\lambda^{2/3}}\langle w_{n+1}C_{n+1}\rangle\langle w_{n-1}\rangle\langle w_{n-2}\rangle\langle w_{n-3}\rangle$$

$$G_n = -a\langle w_{n+2}C_{n+2}\rangle\langle w_{n+1}^2\rangle\langle w_{n-1}\rangle\langle w_{n-2}\rangle\langle w_{n-3}\rangle$$

$$H_n = -b\langle w_{n+1}C_{n+1}\rangle\langle w_{n-1}\rangle\langle w_{n-2}\rangle\langle w_{n-3}\rangle - \frac{c}{\lambda^{2/3}}\langle\frac{C_{n-1}}{w_{n-1}}\rangle\langle w_{n-3}\rangle$$
$$+\frac{b}{\lambda^{4/3}}\langle w_{n-1}C_{n-1}\rangle\langle w_{n-3}\rangle - \frac{c}{\lambda^{4/3}}\langle\frac{C_{n-2}}{w_{n-2}}\rangle$$

$$K_n = -c\langle w_{n-2}\rangle\langle w_{n-3}\rangle$$

$$L_n = -\frac{b}{\lambda^{2/3}}\langle w_{n-2}\rangle\langle w_{n-3}\rangle - \frac{a}{\lambda^{4/3}}\langle w_{n-1}^2\rangle\langle w_{n-2}\rangle\langle w_{n-3}\rangle.$$

In these expressions $S_n = \sin\Delta_n$, $C_n = \cos\Delta_n$ for all n. Note that the resulting closed equations for the P_n's are nonlinear integro-partial differential equations, since the averages in the above expressions are over the P_n's themselves.

If we assume further that all the "spins" ξ_n are identically distributed, i.e. $P_n = P$ for all n, then the above equations for P_n for each n reduce to the same equation for P, after a change in the time-scale by a factor $\frac{k_n^{2/3}}{\lambda}\langle w\rangle^{n-4}$:

$$\partial_t P(\xi) = -\frac{\partial}{\partial w}\left[\overline{W}(\xi)P(\xi)\right] - \frac{\partial}{\partial\Delta}\left[\overline{Z}(\xi)P(\xi)\right], \tag{15}$$

with now

$$\overline{W}(\xi) = (Aw^3 - Bw^2 + Cw) + (Dw^2 - E)\sin\Delta$$

$$\overline{Z}(\xi) = (Fw^2 + Gw + H) - (Kw^{-1} + Lw)\cos\Delta$$

where

$$A = -\frac{a}{\lambda^{2/3}}\langle wS\rangle\langle w\rangle^3$$

$$B = -a\langle wS\rangle\langle w^2\rangle\langle w\rangle^3$$

$$C = b\langle wS\rangle\langle w\rangle^3 - \frac{c}{\lambda^{2/3}}\langle\frac{S}{w}\rangle\langle w\rangle$$

$$D = -\frac{b}{\lambda^{2/3}}\langle w\rangle^2$$

$$E = -c\langle w\rangle^2$$

$$F = \frac{a}{\lambda^{2/3}}\langle wC\rangle\langle w\rangle^3$$

$$G = -a\langle wC\rangle\langle w^2\rangle\langle w\rangle^3$$

$$H = -b\langle wC\rangle\langle w\rangle^3 - \frac{c}{\lambda^{2/3}}\langle\frac{C}{w}\rangle\langle w\rangle + \frac{b}{\lambda^{4/3}}\langle wC\rangle\langle w\rangle - \frac{c}{\lambda^{4/3}}\langle\frac{C}{w}\rangle$$

$$K = -c\langle w\rangle^2$$

$$L = -\frac{b}{\lambda^{2/3}}\langle w\rangle^2 - \frac{a}{\lambda^{4/3}}\langle w^2\rangle\langle w\rangle^2.$$

Using $a+b+c=0$, it is easy to verify that $P_\pm(\xi) = \delta(w-1)\delta(\Delta \mp \frac{\pi}{2})$ are exact time-independent solutions of (15). The presence of these solutions is due to the well-known existence of exact, steady-state "K41" solutions of the SABRA model, of the form $u_n^\pm = \pm iAk_n^{-1/3}$ for any choice of the real constant $A > 0$. The solution u_n^+ has a backward energy transfer to low wavenumbers, while u_n^- has forward transfer to high wavenumbers.

It is interesting that $P_-(\xi)$ is linearly stable under the dynamics (15), while $P_+(\xi)$ is unstable. In fact, the linearization of (15) is

$$\begin{aligned}\partial_t\delta P(\xi) \quad = \quad & -\frac{\partial}{\partial w}\left[\overline{W}_\pm(\xi)\delta P(\xi)\right] - \frac{\partial}{\partial\Delta}\left[\overline{Z}_\pm(\xi)\delta P(\xi)\right] \\ & -\frac{\partial}{\partial w}\left[\delta\overline{W}(\xi)P_\pm(\xi)\right] - \frac{\partial}{\partial\Delta}\left[\delta\overline{Z}_\pm(\xi)P_\pm(\xi)\right]\end{aligned} \qquad (16)$$

with

$$\overline{W}_\pm(\xi) = (A_\pm w^3 - B_\pm w^2 + C_\pm w) + (D_\pm w^2 - E_\pm)\sin\Delta \qquad (17)$$

$$\overline{Z}_\pm(\xi) = -(K_\pm w^{-1} + L_\pm w)\cos\Delta \qquad (18)$$

where

$$A_\pm = \mp\frac{a}{\lambda^{2/3}}, B_\pm = \mp a, C_\pm = \pm\left(b - \frac{c}{\lambda^{2/3}}\right), D_\pm = -\frac{b}{\lambda^{2/3}}, E_\pm = -c,$$

$$F_\pm = G_\pm = H_\pm = 0, K_\pm = -c, L_\pm = -\left(\frac{b}{\lambda^{2/3}} + \frac{a}{\lambda^{4/3}}\right).$$

and $\delta\overline{W}(\xi), \delta\overline{Z}(\xi)$ have the same form as $\overline{W}(\xi), \overline{Z}(\xi)$ but with coefficients $\delta A, \delta B, ...$, etc. that can be obtained by linearizing the corresponding coefficients $A, B, ...$, etc. Now the essential fact is that $\xi_\pm = (1, \pm\frac{\pi}{2})$ is an unstable/stable fixed point of the dynamical system $(\dot{w}, \dot{\Delta}) = (\overline{W}_\pm(\xi), \overline{Z}_\pm(\xi))$, for $\pm$ respectively. This is easily verified directly from the equations (17),(18). For example, the linearization about those fixed points is

$$\left(\begin{array}{cc} \frac{\partial \overline{Z_-}}{\partial w} & \frac{\partial \overline{Z_-}}{\partial \Delta} \\ \frac{\partial \overline{W_-}}{\partial w} & \frac{\partial \overline{W_-}}{\partial \Delta} \end{array}\right)\Bigg|_{\xi=\xi_+} = \left(\begin{array}{cc} \left(1-\frac{1}{\lambda^{2/3}}\right)(a-c) & 0 \\ 0 & \left(1-\frac{1}{\lambda^{4/3}}\right)a + \left(1-\frac{1}{\lambda^{2/3}}\right)b \end{array}\right).$$

$$\left(\begin{array}{cc} \frac{\partial \overline{Z_+}}{\partial w} & \frac{\partial \overline{Z_+}}{\partial \Delta} \\ \frac{\partial \overline{W_+}}{\partial w} & \frac{\partial \overline{W_+}}{\partial \Delta} \end{array}\right)\Bigg|_{\xi=\xi_-} = -\left(\begin{array}{cc} (2a+b)+\frac{2a+5b}{\lambda^{2/3}} & 0 \\ 0 & \left(1-\frac{1}{\lambda^{4/3}}\right)a + \left(1-\frac{1}{\lambda^{2/3}}\right)b \end{array}\right).$$

When $a > 0$ and $b, c < 0$, then we see that both eigenvalues are positive for the $+$ fixed point, and the second eigenvalue is negative for the $-$ fixed point. The first eigenvalue of the $-$ fixed point is also negative when

$$2a + 5b > -(2a+b)\lambda^{2/3},$$

which imposes a condition on the coefficients. For example, with the common parameterization $a = 1, b = -\epsilon, c = \epsilon - 1$, a region in the (ϵ, λ)-plane is selected specified by $2 - 5\epsilon > (\epsilon - 2)\lambda^{2/3}$. Along the curve $\lambda = 1/(1-\epsilon)$ for which the second invariant is helicity-like, the condition is that λ should lie in an interval $(1, \lambda_*)$ with $\lambda_* \approx 2.467$. In particular, the standard case $\lambda = 2, a = 1, b = c = -1/2$ which we simulated in this paper lies in this region. For all parameter values satisfying the above condition ξ_+ is linearly unstable, and ξ_- linearly unstable.

Let then $\mathcal{B}_\delta(\xi_\pm)$ be a disk of radius δ centered at $\xi_\pm$ in the strip $(0, \infty) \times [-\pi, \pi]$ of the (w, Δ)-plane, and let $\mathcal{B}^c_\delta(\xi_\pm)$ be its complement. Since the integral over $\mathcal{B}^c_\delta(\xi_\pm)$ of the second set of terms in (16) vanishes identically, it follows that the perturbation term $\int_{\mathcal{B}^c_\delta(\xi_\pm)} d\xi\ \delta P(\xi)$ satisfies the same equation as it would under the flow of the vector field in (18)-(17). In that case, for the $+$ sign, $\int_{\mathcal{B}^c_\delta(\xi_+)} d\xi\ \delta P(\xi)$ increases in time, whereas for the $-$ sign $\int_{\mathcal{B}^c_\delta(\xi_-)} d\xi\ \delta P(\xi)$ decreases in time, for any $\delta > 0$. We therefore see that solution $P_+(\xi)$ is linearly unstable, whereas the solution $P_-(\xi)$ is linearly stable. In fact, a direct integration of the full nonlinear equation (15) shows that $P_-(\xi)$ is the global attractor for all initial probability density functions $P(\xi)$. See [6].

This argument, given here assuming perfect independence of "spins" at different shellnumbers, can be generalized assuming a correlation of finite range r. In that case, one can develop a similar equation for the r-spin distribution $P_r(\xi_n, \xi_{n+1}, ..., \xi_{n+r})$. It is found in the same manner that a delta-function at the K41 fixed point value ξ_- in all shells is the unique, global attracting solution. For details, see [6].

A.2 Markov Chain Models of Multipliers

In this second appendix we briefly discuss Markov chain models for the multipliers. We shall assume that $\xi_n = (w_n, \Delta_n)$ has statistics derived from a stationary Markov chain with single-shell distribution $P(\xi_n)$ and forward transition probability $T(\xi_{n+1}|\xi_n),\ n = 1, 2, 3, ...$ Because of the stationarity assumption, these functions do not depend upon shellnumber n. We first discuss the relation between the structure-function scaling exponents and the eigenvalues of certain "transfer matrices". This is a particular example of the connection discussed in the text between scaling exponents and free energy functions. We next discuss the scaling properties of the mean fluxes of conserved quantities. In particular, we show how Markov chain models can yield a joint cascade of both energy and helicity.

To calculate the absolute structure functions, it is easiest to consider the time-reversed Markov chain, with backward transition probability

$$\widetilde{T}(\xi_n|\xi_{n+1}) = \frac{T(\xi_{n+1}|\xi_n)P(\xi_n)}{P(\xi_{n+1})}.$$

Then, using $\rho_n^p = \prod_{k=1}^n w_k^p = \prod_{k=1}^n e^{p\sigma_k}$,

$$\begin{aligned}\langle \rho_n^p \rangle &= \int d\xi_{n+1}\ P(\xi_{n+1}) \int d\xi_n\ e^{p\sigma_n} \widetilde{T}(\xi_n|\xi_{n+1}) \cdots \int d\xi_1\ e^{p\sigma_1} \widetilde{T}(\xi_1|\xi_2) \\ &= \int d\xi_{n+1}\ P(\xi_{n+1}) \int d\xi_1\ T_{(p)}^n(\xi_1|\xi_{n+1}) \qquad (19)\end{aligned}$$

where the "transfer matrix" $T_{(p)}$ is defined by

$$T_{(p)}(\xi_n|\xi_{n+1}) := e^{p\sigma_n} \widetilde{T}(\xi_n|\xi_{n+1})$$

and $T_{(p)}^n$ is its n-fold convolution or matrix product. For large n, we have asymptotically that

$$T_{(p)}^n(\xi_1|\xi_{n+1}) \sim \mu_{(p)}^n R_{(p)}(\xi_1) L_{(p)}(\xi_{n+1}) \qquad (20)$$

where $\mu_{(p)}$ is the principal eigenvalue of $T_{(p)}$ and $R_{(p)}, L_{(p)}$ are corresponding right and left eigenfunctions. By the Perron-Frobenius theorem, these eigenvalues are real and non-negative, and the left and right eigenfunctions may also be chosen to be non-negative. Thus, we find that

$$\langle \rho_n^p \rangle \sim \mu_{(p)}^n \langle L_{(p)} \rangle \overline{R}_{(p)}$$

for $n \to \infty$, where $\langle L_{(p)} \rangle = \int d\xi\ P(\xi) L_{(p)}(\xi)$ and $\overline{R}_{(p)} = \int d\xi\ R_{(p)}(\xi)$. In this way, we obtain the relationship between the scaling exponents of structure functions and eigenvalues of the transfer matrices as

$$\zeta_p = \frac{p}{3} - \log_\lambda(\mu_{(p)}).$$

Exact moment constraints require that $\mu_{(3)} = 1$, so that $\zeta_3 = 1$ within a Markov chain model. For details, see [6].

We now consider the asymptotics of mean fluxes of the inviscid invariants. In the SABRA model, the energy flux is given by

$$\Pi_n^E = -ak_n\mathcal{T}_{n+1} + ck_{n-1}\mathcal{T}_n$$

and the helicity flux by

$$\Pi_n^H = -2a\left(\frac{a}{c}\right)^n (k_n\mathcal{T}_{n+1} - k_{n-1}\mathcal{T}_n).$$

Here $\mathcal{T}_n$ is a triple velocity product

$$\mathcal{T}_n = \mathrm{Im}(u_{n+1}^* u_n u_{n-1}) = \frac{1}{k_n}\rho_{n+1}\rho_n\rho_{n-1}\sin\Delta_{n+1}$$

In terms of the scale-local variables $w_n, \Delta_n,\ n = 1, 2, 3, ...$

$$\Pi_n^E = \frac{1}{\lambda}\left[-a\cdot w_{n+2}\sin\Delta_{n+2}\cdot w_{n+1}^2\cdot w_n^3 + c\cdot w_{n+1}\sin\Delta_{n+1}\cdot w_n^2\right]\prod_{k=1}^{n-1} w_k^3$$

and

$$\Pi_n^H = \frac{-2a}{\lambda}\left(\frac{a}{c}\right)^n\left[w_{n+2}\sin\Delta_{n+2}\cdot w_{n+1}^2\cdot w_n^3 - w_{n+1}\sin\Delta_{n+1}\cdot w_n^2\right]\prod_{k=1}^{n-1} w_k^3.$$

We now evaluate the mean values of these flux variables. If $P_2(\xi_{n+2}, \xi_{n+1}) = \widetilde{T}(\xi_{n+1}|\xi_{n+2})P(\xi_{n+2})$ is the joint distribution of "spins" ξ_{n+2}, ξ_{n+1} at two successive shells, then

$$\begin{aligned}\langle\Pi_n^E\rangle &= \frac{1}{\lambda}\\ &\left[-a\int d\xi_{n+2}\int d\xi_{n+1}\ P_2(\xi_{n+2},\xi_{n+1})\cdot w_{n+2}\sin\Delta_{n+2}\cdot w_{n+1}^2\cdot\int d\xi_1\ T_{(3)}^n(\xi_1|\xi_{n+1})\right.\\ &\left.+c\int d\xi_{n+1}\int d\xi_n\ P_2(\xi_{n+1},\xi_n)\cdot w_{n+1}\sin\Delta_{n+1}\cdot w_n^2\int d\xi_1\ T_{(3)}^{n-1}(\xi_1|\xi_n)\right]\end{aligned} \tag{21}$$

and

$$\begin{aligned}\langle\Pi_n^H\rangle &= -\frac{2a}{\lambda}\left(\frac{a}{c}\right)^n\\ &\left[\int d\xi_{n+2}\int d\xi_{n+1}\ P_2(\xi_{n+2},\xi_{n+1})\cdot w_{n+2}\sin\Delta_{n+2}\cdot w_{n+1}^2\cdot\int d\xi_1\ T_{(3)}^n(\xi_1|\xi_{n+1})\right.\\ &\left.-\int d\xi_{n+1}\int d\xi_n\ P_2(\xi_{n+1},\xi_n)\cdot w_{n+1}\sin\Delta_{n+1}\cdot w_n^2\int d\xi_1\ T_{(3)}^{n-1}(\xi_1|\xi_n)\right]\end{aligned} \tag{22}$$

To calculate the mean energy flux asymptotically for large n, it suffices to use the previous asymptotic expansion (20) for $p = 3$. This gives

$$\langle\Pi_n^E\rangle \sim -\frac{1}{\lambda}(a\mu_{(3)} - c)\mu_{(3)}^{n-1}\langle w_2\sin\Delta_2\cdot w_1^2 L_{(3)}(w_1,\Delta_1)\rangle\overline{R}_{(3)}$$

as $n \to \infty$. The only way that the energy flux can be asymptotically constant, i.e. independent of shellnumber n, is if $\mu_{(3)} = 1$. This is another argument for that constraint, independent of that given in [6]. In that case, we obtain finally that

$$\langle \Pi_n^E \rangle \sim -\frac{1}{\lambda}(a-c)\langle w_2 \sin\Delta_2 \cdot w_1^2 L_{(3)}(w_1,\Delta_1)\rangle \overline{R}_{(3)}$$

asymptotically for $n \to \infty$.

However, for helicity flux, the contribution from the leading-order term in the asymptotic expansion (20) gives zero identically, because

$$\langle w_{n+2}\sin\Delta_{n+2}\cdot w_{n+1}^2 L_{(3)}(w_{n+1},\Delta_{n+1})\rangle = \langle w_{n+1}\sin\Delta_{n+1}\cdot w_n^2 L_{(3)}(w_n,\Delta_n)\rangle,$$

by stationarity of the Markov chain. Thus, a non-vanishing contribution is obtained only from the next order in the asymptotic expansion for large n,

$$T_{(3)}^n(\xi_n|\xi_{n+1}) \sim \mu_{(3)}^n R_{(3)}(\xi_1)L_{(3)}(\xi_{n+1}) + \mu_{(3)}'^n R_{(3)}'(\xi_1)L_{(3)}'(\xi_{n+1})$$

where $\mu_{(3)}'$ is the subleading eigenvalue of $T_{(3)}$ (i.e. the complex eigenvalue with next largest magnitude $|\mu_{(3)}'|$ after $|\mu_{(3)}|$), and $R_{(3)}', L_{(3)}'$ the corresponding right and left eigenfunctions. In that case,

$$\langle \Pi_n^H \rangle \sim -\frac{2a}{\lambda}\left(\frac{a}{c}\mu_{(3)}'\right)^n \left(1 - \frac{1}{\mu_{(3)}'}\right) \langle w_2 \sin\Delta_2 \cdot w_1^2 L_{(3)}'(w_1,\Delta_1)\rangle \overline{R}_{(3)}'$$

This flux is constant precisely when $\mu_{(3)}' = c/a$. In that case,

$$\langle \Pi_n^H \rangle \sim \frac{2a}{\lambda c}(a-c)\langle w_2 \sin\Delta_2 \cdot w_1^2 L_{(3)}'(w_1,\Delta_1)\rangle \overline{R}_{(3)}'$$

asymptotically for $n \to \infty$.

When $c/a < 0$, a non-zero helicity flux is ruled out in an independent multiplier model by the realizability inequality $\langle w^3 \rangle > 0$. However, in a Markov chain model, there is no such constraint, because the subleading eigenvalue $\mu_{(3)}'$ may easily be negative. As a concrete example, consider the Markov chain with single-shell distribution

$$P(\xi_n) = Ce^{-\beta\cdot\sin\Delta_n} P_W(w_n),$$

where $P_W(w_n)$ is any density on the interval $(0,\infty)$, and with transition probability density

$$\widetilde{T}(\xi_n|\xi_{n+1}) = P(\xi_n)\cdot\left[1 + \left(\frac{c}{a}\right)\mathrm{sgn}(\cos\Delta_n)\mathrm{sgn}(\cos\Delta_{n+1})\right].$$

The latter kernel is non-negative when $|c/a| < 1$. In fact, it is a rank 2 operator and can be written as

$$\widetilde{T}(\xi_n|\xi_{n+1}) = R(\xi_n)L(\xi_{n+1}) + \left(\frac{c}{a}\right) R'(\xi_n)L'(\xi_{n+1})$$

where

$$R(\xi_n) = P(\xi_n), \quad L(\xi_n) = 1, \quad R'(\xi_n) = P(\xi_n)\mathrm{sgn}(\cos\Delta_n), \quad L'(\xi_n) = \mathrm{sgn}(\cos\Delta_n).$$

It is easily checked that these four vectors are a bi-orthonormal set, and can be completed to a bi-orthonormal basis. It follows directly that

$$\int d\xi_n\ \widetilde{T}(\xi_n|\xi_{n+1}) = 1,$$

so that $\widetilde{T}$ is a transition probability density, as claimed. Also P is its invariant distribution, because

$$\int d\xi_{n+1}\ \widetilde{T}(\xi_n|\xi_{n+1})P(\xi_{n+1}) = P(\xi_n).$$

Furthermore, if $\int dw\ w^3 P_W(w) = 1$, then by construction the only two non-vanishing eigenvalues of $T_{(3)}(\xi_n|\xi_{n+1}) = w_n^3\widetilde{T}(\xi_n|\xi_{n+1})$ are $\mu_{(3)} = 1$ and $\mu'_{(3)} = c/a$, with

$$R_{(3)}(\xi_n) = w_n^3 P(\xi_n), \quad L_{(3)}(\xi_n) = 1$$

$$R'_{(3)}(\xi_n) = w_n^3 P(\xi_n)\mathrm{sgn}(\cos\Delta_n), \quad L'_{(3)}(\xi_n) = \mathrm{sgn}(\cos\Delta_n).$$

This example is not completely realistic as a statistical model for the shell dynamics, because the amplitude multipliers $w_n,\ n = 1, 2, 3, ...$ form an i.i.d. sequence. Hence, it is not consistent with the long power-law tail $\sim w^{-3}$ which we have observed in the numerical simulation. It is also does not have a non-vanishing flux of helicity because, unfortunately, the expectations $\langle w_2 \sin\Delta_2 \cdot w_1^2 L'_{(3)}(w_1, \Delta_1)\rangle = \overline{R}'_{(3)} = 0$ in this model. However, it illustrates how it is possible to get the subleading eigenvalue $c/a < 0$ within a realizable Markov chain model.

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